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Probability of large deviations of sums of random processes from Orlicz space

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Abstract. This paper is devoted to the accuracy and reliability estimation (in uniform metrics) of calculation of improper integrals depending on a parameter t, using the Monte Carlo method. For this, estimates for the probability of deviation in the uniform metric of sums of independent identically distributed fields, which belong to Orlicz spaces, were found.

Keywords. Orlicz space of random variables, $\mathcal{C}$-function, Monte Carlo method, random process.

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1 Introduction

The exponential bounds for the probability deviations of normalized sums from the mean in space $\mathbf{C}(\mathbf{T})$ for independent equally distributed random fields were studied in the paper [6]. Random fields, for which Bernstein's condition takes place, were studied in this paper. In this paper the problem, proposed in [6] is examined in a more general setting, namely, we deal with the casual fields from Orlicz spaces of random variables. It should be noted that in case which is considered in [6], the estimates are more accurate than those obtained in our study.

The results of our research may be used for the determination of accuracy in $\mathbf{C}(\mathbf{T})$ and reliability of improper integrals calculation using Monte Carlo method. That is, for calculation of the integral $\int \cdots \int_{R^d} f(\vec{x}) d\vec{x}$ using the Monte Carlo method.

The paper consists of three sections. The main definitions and notions for random processes from Orlicz spaces of random variables are presented in the first section. The second section is devoted to integral calculation with given accuracy and reliability, and calculation of integrals depending on a parameter is presented in the third section.

2 Random processes from Orlicz space of random variables

Definition 2.1 ([3]). A continuous even convex function $U = \{U(x), x \in \mathcal{R}\}$ is called a $\mathcal{C}$-function if $U(x)$ is monotonically increasing function for $x > 0$ and $U(0) = 0$.

Example 2.2. Simple examples of $\mathcal{C}$-function are as follows:

1. $U(x) = a|x|^\alpha, x \in \mathcal{R}, a > 0, \alpha \geq 1$;
2. $U(x) = c(\exp\{a|x|^\alpha\} - 1), x \in \mathcal{R}, c > 0, a > 0, \alpha \geq 1$.

Let $\{\Omega, \Im, P\}$ be a standard probability space.

Definition 2.3 ([3]). Let U be an arbitrary $\mathcal{C}$-function. The Orlicz space of random variables $L_U(\Omega)$ is defined as a family of random variables, where for each $\xi \in L_U(\Omega)$ there exists a constant $r_\xi > 0$ such that

$$\mathsf{E}U\left(\frac{\xi}{r_\xi}\right) < \infty.$$

The space $L_U(\Omega)$ is a Banach space with respect to the norm $\|\xi\|_U = \inf\{r > 0; \mathsf{E}U(\frac{\xi}{r}) \leq 1\}$ [3] (Luxemburg norm).

Example 2.4. Suppose that $U(x) = |x|^p, x \in \mathcal{R}, p \geq 1$. Then $L_U(\Omega)$ is the space $L_p(\Omega)$ and the Luxemburg norm $\|\xi\|_U$ coincides with the norm $\|\xi\|_p = (\mathsf{E}|\xi|^p)^{1/p}$.

Definition 2.5 ([3]). Suppose that $\varphi = (\varphi(x), x \in \mathcal{R})$ is an arbitrary $\mathcal{C}$-function. The Orlicz space generated by the $\mathcal{C}$-function $U(x) = \exp\{\varphi(x)\} - 1, x \in \mathcal{R}$, is called an Orlicz space of exponential type.

We denote this space by $\mathrm{Exp}_\varphi(\Omega)$ and the norm in the space $\mathrm{Exp}_\varphi(\Omega)$ by $\|\cdot\|_{E\varphi}$.

Lemma 2.6 ([3]). *Suppose that $\xi \in L_U(\Omega)$ and $\|\xi\|_U > 0$. Then we have*

$$P\{|\xi| \geq x\} \leq \frac{1}{U(x/\|\xi\|_U)} \tag{2.1}$$

for all $x > 0$.

Definition 2.7 ([3]). We say that a $\mathcal{C}$-function U satisfies g-condition if there exist constants $z_0 \geq 0$, $K > 0$ and $A > 0$ such that the inequality

$$U(x)U(y) \leq AU(Kxy)$$

holds for all $x \geq z_0$, $y \geq z_0$.

Example 2.8. The function $U(x) = a|x|^\alpha$, $x \in \mathcal{R}$, $a > 0$, $\alpha \geq 1$, satisfies g-condition with $K = 1$, $A = a$ and $z_0 = 0$.

The $\mathcal{C}$-function $U(x) = \exp\{\varphi(x)\} - 1$, $x \in \mathcal{R}$, where $\varphi = (\varphi(x), x \in \mathcal{R})$ is an arbitrary $\mathcal{C}$-function, satisfies g-condition with $K = 1$, $A = 1$, $z_0 = 2$ (if $\varphi(x) = |x|^\alpha, \alpha \geq 1,$ then $z_0 = 2^{1/\alpha}$).

Lemma 2.9. *Let m be a constant. Then $m \in L_U(\Omega)$ for any Orlicz space $L_U(\Omega)$ and $\|m\|_U = \frac{|m|}{U^{(-1)}(1)}$.*

Lemma 2.9 is evident.

Lemma 2.10. *Let $\xi \in L_U(\Omega)$. Then there exists such constant d_U that $E|\xi| \leq d_U\|\xi\|_U$.*

This lemma is a consequence of the Theorem 2.3.2 from [3].

Example 2.11. For $U(x) = |x|^p, p \geq 2$, we have $d_U = 1$. For $U(x) = \exp\{\varphi(x)\}-1$, where $\varphi(x)$ is an $\mathcal{N}$-function (that is such $\mathcal{C}$-function that $\frac{\varphi(x)}{x} \to 0$ as $x \to 0$, $\frac{\varphi(x)}{x} \to \infty$ as $x \to \infty$), we have $d_U = \frac{2}{[\varphi^*]^{(-1)}(1)}$, where $[\varphi^*](x)$ is the inverse function of $(\varphi^*(x), x \in \mathcal{R})$, $\varphi^*(x) = \sup_{y\in\mathcal{R}}(xy - \varphi(x))$ is the Young–Fenchel transform of the function $\varphi(x)$ (see Lemma 2.3.3 from [3]).

For example, if $\varphi(x) = \frac{|x|^\alpha}{\alpha}, \alpha > 1$, then $\varphi^*(x) = \frac{|x|^\beta}{\beta}$, where $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, $\varphi^{*(-1)}(x) = (x\beta)^{1/\beta}$ and $\varphi^{*(-1)}(1) = (\beta)^{1/\beta}$. If $\alpha = 2$, then $\beta = 2$ and $d_U = \sqrt{2}$. If $\alpha = 4$, then $\beta = \frac{4}{3}$ and $d_U = \frac{3^{3/4}}{4^{1/4}}$.

Definition 2.12. An Orlicz space $L_U(\Omega)$ has the property **H** if for any centered independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ from $L_U(\Omega)$ the following inequality holds

$$\Big\| \sum_{k=1}^{n} \xi_i \Big\|_U^2 \leq C_U \sum_{k=1}^{n} \|\xi_i\|_U^2$$

where C_U is some absolute constant.

The examples of spaces, which satisfy the condition **H**, are shown below:

- spaces $L_p(\Omega)$, $p \geq 2$, where $C_U = C_p = \sqrt{2}(\Gamma(p+1)/2\sqrt{\pi})^{1/p}$ [7];
- spaces $L_U(\Omega)$, where $U(x)$ such $\mathcal{C}$-functions that there exist $p > q \geq 2$, for which $U(\sqrt[q]{x})$ is convex and $U(\sqrt[p]{x})$ is concave, and $C_U = 2B_p$ [2]. Value B_p can be found in [8], p. 341;

- Orlicz spaces generated by $\mathcal{C}$-function $U(x) = \exp\{|x|^\alpha\}$, where $x > x_0 > 0$, $0 < \alpha \le 1$ [1];
- Orlicz spaces generated by $\mathcal{C}$-function $U(x) = \exp\{|x|^\alpha\} - 1$, where $1 \le \alpha \le 2$ (see [1] and book [3]). It is shown in the paper [5] that for $\alpha \ge 2$ these spaces do not possess **H** condition.

Definition 2.13. Let $X = \{X(t), t \in \mathbf{T}\}$ be a random process, $\mathbf{T}$ be a nonempty parametric set. The process X belongs to an Orlicz space $L_U(\Omega)$, if for all $t \in \mathbf{T}$ the random variable $X(t) \in L_U(\Omega)$.

Let $\rho(t,s) = \|X(t) - X(s)\|_U$ be a pseudometric generated in $\mathbf{T}$ by the process $X = \{X(t), t \in \mathbf{T}\} \in L_U(\Omega)$. Consider a pseudometric space $(\mathbf{T}, \rho)$. Let $N_\rho(v)$ be the metric massiveness of the $(\mathbf{T}, \rho)$, that is the smallest number of elements in an v-covering of the set $\mathbf{T}$. That is, the covering of $\mathbf{T}$ by balls of radius at most v.

Theorem 2.14. *Let $X = \{X(t), t \in \mathbf{T}\}$ be a random process from a space $L_U(\Omega)$, where $\mathcal{C}$-function U satisfies* g*-condition. The process X is separable on $(\mathbf{T}, \rho)$, $\sup_{t\in\mathbf{T}} \|X(t)\|_U < \infty$, $\varepsilon_0 = \sup_{t,s\in\mathbf{T}} \rho(t,s)$.*

If the following condition holds

$$\int_0^{\varepsilon_0} U^{(-1)}(N_\rho(\varepsilon))d\varepsilon < \infty, \tag{2.2}$$

then

1. $$\sup_{t\in\mathbf{T}} |X(t)| \in L_U(\Omega);$$

2. $$\|\sup_{t\in\mathbf{T}} |X(t)|\,\|_U \le B = \inf_{t\in\mathbf{T}} \|X(t)\|_U + \inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\varepsilon_0} \nu(N_\rho(\varepsilon))d\varepsilon \tag{2.3}$$

   where

   $$\nu(n) = \begin{cases} n, & \text{if } n \le U(z_0); \\ K(1+U(z_0))DU^{(-1)}(n), & \text{if } n > U(z_0), \end{cases}$$

   $D = \max\{1, A\}$*, A, K and z_0 are the constants from* g*-condition.*

3. *For any $\varepsilon > 0$*

   $$P\{\sup_{t\in\mathbf{T}} |X(t)| \ge \varepsilon\} \le \frac{1}{U\left(\frac{x}{B}\right)}.$$

Proof. Theorem 2.14 is a modification of the Corollary 3.3.1 from the book [3]. □

Theorem 2.15. *Let* $(\mathbf{T}, w)$ *be a compact metric space,* $X = \{X(t), t \in \mathbf{T}\}$ *be a random process from Orlicz space* $L_U(\Omega)$*. The condition* g *holds for* U*. Let* X *be separable on* $(\mathbf{T}, w)$ *and there exist a function* $\sigma = \{\sigma(h),\ 0 \le h \le \sup_{t,s \in \mathbf{T}} \rho(t,s)\}$ *such that* $\sigma(h)$ *is monotonically increasing, continuous and*

$$\sup_{\rho(t,s)\le h} \|X(t) - X(s)\|_U \le \sigma(h).$$

Let $N_w(u)$ *be the metric massiveness of the space* $(\mathbf{T}, w)$*. Suppose that*

$$\int_0^{\delta_0} U^{(-1)}(N_w(\sigma^{(-1)}(u)))du < \infty,$$

where $\delta_0 = \sigma(\sup_{t,s \in \mathbf{T}} w(t,s))$*,* $\sigma^{(-1)}(u)$ *is the inverse function of* $\sigma(\cdot)$*.*

Then

1. $$\sup_{t\in\mathbf{T}} |X(t)| \in L_U(\Omega),$$

2. $$\|\sup_{t\in\mathbf{T}} |X(t)|\,\|_U \le \widetilde{B}$$

$$= \inf_{t\in\mathbf{T}} \|X(t)\|_U + \inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\delta_0} \nu(N_w(\sigma^{(-1)}(u)))du.$$

Proof. This theorem follows from the Theorem 2.14, since separability of the process X on $(\mathbf{T}, w)$ follows from separability of $X(t)$ on $(\mathbf{T}, \rho)$ and the following inequality holds:

$$N_\rho(u) \le N_w(\sigma^{-1}(u))$$ □

Consider the space R^d with the metric $m(\vec{x}, \vec{y}) = \max_{1\le i\le d} |x_i - y_i|$.

Corollary 2.16. *Let* $\mathbf{T}$ *be the cube* $\{0 \le x_i < T, i = \overline{1,d}\}, T > 0$*. Then in the Theorem* 2.15

$$N_w(u) \le \left(\frac{T}{2u} + 1\right)^d$$

and we have such inequalities:

$$\widetilde{B} \le \widehat{B} = \inf_{t\in\mathbf{T}} \|X(t)\|_U + \inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\delta_0} \nu\left(\left(\frac{T}{2\sigma^{(-1)}(u)} + 1\right)^d\right)du.$$

3 The error of computing of integrals by the Monte Carlo method

Theorem 3.1. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent identically distributed and belong to an Orlicz space $L_U(\Omega)$. The Orlicz space $L_U(\Omega)$ has the property* **H**. *Let $y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\xi_i - \mathbf{I})$, where $\mathbf{I} = E\xi_i$.*

Then for any $\varepsilon > 0$ the following inequality holds

$$P\{|y_n| > \varepsilon\} \le \frac{1}{U\left(\frac{\varepsilon}{R}\right)} \tag{3.1}$$

where $R = \|\xi_1 - \mathbf{I}\|_U \sqrt{C_U}$, C_U is the constant from Definition 2.12.

Proof. It follows from Definition 2.12 that $\|y_n\|_U^2 = \|\frac{1}{\sqrt{n}} \sum_{i=1}^n (\xi_i - \mathbf{I})\|_U^2 = \frac{1}{n}\| \sum_{i=1}^n (\xi_i - \mathbf{I})\|_U^2 \le \frac{1}{n} C_U \sum_{i=1}^n \|\xi_i - \mathbf{I}\|_U^2 = C_U \|\xi_1 - \mathbf{I}\|_U^2$. Now (3.1) follows from Lemma 2.6. □

Corollary 3.2. *Let assumptions of the Theorem* 3.1 *hold. Then for any $\varepsilon > 0$ the following inequality holds:*

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^n \xi_i - \mathbf{I}\right| > \varepsilon\right\} \le \frac{1}{U\left(\frac{\sqrt{n}\varepsilon}{R}\right)} \tag{3.2}$$

Proof. $\frac{1}{n}\sum_{i=1}^n \xi_i - \mathbf{I} = \frac{1}{n}\sum_{i=1}^n (\xi_i - \mathbf{I}) = \frac{1}{\sqrt{n}} y_n$. Therefore

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^n \xi_i - \mathbf{I}\right| > \varepsilon\right\} = P\left\{\frac{1}{\sqrt{n}}|y_n| > \varepsilon\right\} = P\left\{|y_n| > \sqrt{n}\varepsilon\right\} \le \frac{1}{U\left(\frac{\sqrt{n}\varepsilon}{R}\right)}.$$

□

Let $\{\mathcal{S}, \mathcal{A}, \mu\}$ be a measurable space, μ be a σ-finite measure and $p(s) \ge 0$, $s \in \mathcal{S}$ be a such measurable function that $\int_{\mathcal{S}} p(s)d\mu(s) = 1$. Let $m(A)$, $A \in \mathcal{A}$ be the measure $m(A) = \int_A p(s)d\mu(s)$. $m(A)$ is a probability measure and the space $\{\mathcal{S}, \mathcal{A}, m\}$ is a probability space.

Let $f(s)$ be a measurable function on $\{\mathcal{S}, \mathcal{A}, \mu\}$. Consider $\int_{\mathcal{S}} f(s)p(s)d\mu(s) = \mathcal{I}$. Suppose, that this integral exist.

Remark 3.3. We can consider the integral of the form $\int_{\mathcal{S}} \varphi(s)d\mu(s)$. Then if $p(s) \ge 0$ is a probability density function in the space $\{\mathcal{S}, \mathcal{A}, \mu\}$, then

$$\int_{\mathcal{S}} \varphi(s)d\mu(s) = \int_{\mathcal{S}} \frac{\varphi(s)}{p(s)} p(s)d\mu(s) = \int_{\mathcal{S}} f(s)p(s)d\mu(s),$$

where $f(s) = \frac{\varphi(s)}{p(s)}$.

We can consider $f(s) = f$ as random variables on $\{\mathcal{S}, \mathcal{A}, m\}$ and

$$\int_{\mathcal{S}} f(s)p(s)d\mu(s) = \int_{\mathcal{S}} f(s)dm(s) = \mathsf{E}f.$$

Let ξ_i, $i = 1, \dots, n$ be independent copies of f, $Z_n = \frac{1}{n}\sum_{i=1}^{n} \xi_i$. Then $Z_n \to \mathsf{E}\xi_1 = \mathcal{I}$ with probability one. We can consider Z_n as an estimate of $\mathcal{I}$.

Definition 3.4. Z_n approaches $\mathcal{I}$ with reliability $1-\delta$ $(0 < \delta < 1)$ and accuracy $\varepsilon > 0$ if the following inequality holds:

$$P\{|Z_n - \mathcal{I}| > \varepsilon\} \le \delta. \tag{3.3}$$

Theorem 3.5. *Let random variable f belong to $L_U(\Omega)$, where the Orlicz space $L_U(\Omega)$ has the property* **H** *with the constant C_U. Then Z_n approaches $\mathcal{I}$ with reliability $1-\delta$ and accuracy ε if the following inequality holds:*

$$n \ge \left(\frac{RU^{(-1)}\left(\frac{1}{\delta}\right)}{\varepsilon}\right)^2, \tag{3.4}$$

where $R = \|f - \mathcal{I}\|_U \sqrt{C_U}$.

Proof. It follows from Corollary 3.2 that $P\{|Z_n - \mathcal{I}| > \varepsilon\} \le (U(\frac{\sqrt{n}\varepsilon}{R}))^{-1}$. The assertion of this theorem holds if $(U(\frac{\sqrt{n}\varepsilon}{R}))^{-1} \le \delta$, that is if (3.4) holds. □

Remark 3.6. Prove that

$$\|f - \mathcal{I}\|_U \le \left(1 + \frac{d_U}{U^{(-1)}(1)}\right) \|f(u)\|_U\,. \tag{3.5}$$

Since $\|f - \mathcal{I}\|_U \le \|f\|_U + \|\mathcal{I}\|_U$ then it follows from Lemma 2.9 that $\|\mathcal{I}\|_U < \frac{|\mathcal{I}|}{U^{(-1)}(1)}$ and from Lemma 2.10 that $|\mathcal{I}| \le d_U \|f\|_U$. Therefore (3.5) holds true.

Example 3.7. Usually Monte Carlo method is used for multiple integrals calculation, but for better understanding let us consider an integral of one variable function. If $\int_{-\infty}^{+\infty} f(x)\exp\{\frac{-x^2}{2a^2} - bx\}dx = I$, $a > 0$, $|f(x)| < 1$, then $I = \sqrt{2\pi}a\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{+\infty} f(x)\exp\{\frac{-x^2}{2a^2} - bx\}dx$. Let denote $J = \mathsf{E}\exp\{-\xi b\}$, where ξ is $N(0, a^2)$, $\eta_i = f(\xi_i)\exp\{-\xi_i b\}$, ξ_i – independent copies of random variable ξ. Let $J_n = \frac{1}{n}\sum_{i=1}^{n}\eta_i = \frac{1}{n}\sum_{i=1}^{n} f(x_i)\exp\{-\xi_i b\}$. The estimation for

I will be $I_n = \sqrt{2\pi} a J_n$. Let put $U(x) = |x|^p$, $p \geq 2$, then according to the remark 3.6 to Theorem 3.5 and value C_U for $L_U(\Omega)$ spaces we receive:

$$R = \|\eta\|_p \, 2\sqrt{\sqrt{2}(\Gamma(p+1)/2\sqrt{\pi})^{1/p}},$$

where $\|\eta\|_p^p = \mathsf{E}\,|f(x)|^p\,(\exp\{-\xi b\})^p \leq \mathsf{E}\exp\{-\xi bp\} = \exp\{\frac{p^2b^2a^2}{2}\}$.

According to the equation (3.4) integral I will be computed with accuracy ε and reliability δ, when the following inequality holds:

$$n \geq \frac{a^2 2\pi R^2}{\varepsilon^2 \delta^{2/p}}.$$

As the last inequality is true for $p \geq 2$, in order to find minimal n it is necessary to minimize the right part on p, i.e. to find approximate value of minimum. According to the Stirling formula $\Gamma(p) \cong \exp\{-p\} p^{p-1/2}(2\pi)^{1/2}$,

$$\frac{R^2}{\delta^{2/p}} \cong \frac{4\sqrt{2}\exp\{pb^2a^2\}p(p/2)^{1/2p}}{\delta^{2/p}},$$

at $p \cong \frac{2(-ln\delta)}{1+\sqrt{1+4a^2b^2(-ln\delta)}}$.

The rate of convergence Z_n to $\mathcal{I}$

Theorem 3.8. *Let assumptions of the Theorem* 3.5 *hold. Then with probability one for sufficiently large n*

$$|Z_n - \mathcal{I}| \leq \frac{R}{\sqrt{n}} U^{(-1)}\left(\frac{1}{\delta_n}\right)$$

where $\delta_n > 0$ *a such sequence that* $\sum_{n=1}^{\infty} \delta_n < \infty$.

Proof. This theorem follows from Borel–Cantelli lemma. Indeed it follows from the Corollary 3.2 that

$$P\left\{|Z_n - \mathcal{I}| > \frac{R}{\sqrt{n}} U^{(-1)}\left(\frac{1}{\delta_n}\right)\right\} \leq \left(U\left(\frac{\sqrt{n}}{R}\frac{R}{\sqrt{n}} U^{(-1)}\left(\frac{1}{\delta_n}\right)\right)\right)^{-1} = \delta_n.$$

□

Example 3.9. If $U(x) = |x|^p$, $p \geq 2$ and $\delta_n = \frac{1}{n^{1+\kappa}}$, $\kappa > 0$ then for sufficiently large n: $|Z_n - \mathcal{I}| \leq \frac{R}{n^{1/2-1/p-\kappa/p}}$ $(\frac{1+\kappa}{p} < \frac{1}{2})$. If $U(x) = \exp\{|x|^\alpha\} - 1$, $1 \leq \alpha \leq 2$ and $\delta_n = \frac{1}{n^{1+\kappa}}$, $\kappa > 0$ then for sufficiently large n: $|Z_n - \mathcal{I}| \leq \hat{R}\frac{1}{n^{1/2}}(\ln n)^{1/\alpha}$, where $\hat{R}$ is some constant.

4 The error of the Monte Carlo calculation of an integral depending on a parameter

Theorem 4.1. *Let $Y = \{Y(t), t \in \mathbf{T}\}$ be a random process. Y belongs to an Orlicz space $L_U(\Omega)$ such that the condition* g *holds for U and $L_U(\Omega)$ has the property* **H** *with the constant C_U.*

Let $(\mathbf{T}, w)$ be a compact metric space and Y be separable process on $(\mathbf{T}, w)$, $N_w(u)$ be the metric massiveness. There exist a continuous and increasing function $\sigma = \{\sigma(h),\ \ 0 \le h \le \sup_{t,s\in\mathbf{T}} \rho(t,s)\}$, that

$$\sup_{\rho(t,s)\le h} \|Y(t) - Y(s)\|_U \le \sigma(h) \tag{4.1}$$

and

$$\int_0^{\delta_0} U^{(-1)}(N_w(\sigma^{(-1)}(u)))du < \infty. \tag{4.2}$$

Let $X(t) = Y(t) - m(t)$, where $m(t) = \mathsf{E}X(t)$ and $X_k(t)$ be independent copies of $X(t)$, $S_n(t) = \frac{1}{\sqrt{n}}\sum_{k=1}^n X_k(t)$. Then for all $\varepsilon > 0$ the following inequality holds

$$P\{\sup_{t\in\mathbf{T}} |S_n(t)| > \varepsilon\} \le \frac{1}{U\left(\frac{\varepsilon}{B(t_0,\theta)}\right)}, \tag{4.3}$$

where t is any point from $\mathbf{T}$, $0 < \theta < 1$,

$$B(t_0, \theta) = \|X(t_0)\|_U + \frac{1}{\theta(1-\theta)} \int_0^{\delta_0\theta} \nu(N_w(\sigma_1^{(-1)}(u))du, \tag{4.4}$$

where $\sigma_1(h) = (1 + \frac{d_U}{U^{(-1)}(1)})\sigma(h)$, d_U is the constant from Lemma 2.10, $\delta_0 = \sigma_1(\sup_{t,s\in\mathbf{T}} \rho(t,s))$, $\nu(n)$ defined in Theorem 2.14.

Proof. It follows from the Definition 2.12 that

$$\begin{aligned} \|S_n(t) - S_n(s)\|_U^2 &= \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n (X_k(t) - X_k(s)) \right\|_U^2 \\ &\le C_U \frac{1}{n} \sum_{k=1}^n \|X_k(t) - X_k(s)\|_U^2 = C_U \|X(t) - X(s)\|_U^2. \end{aligned} \tag{4.5}$$

It follows from Lemma 2.9 and Lemma 2.10 that

$$\begin{aligned}\|X(t)-X(s)\|_U &= \|y(t)-y(s)-(m(t)-m(s))\|_U \\ &\le \|y(t)-y(s)\|_U + \frac{1}{U^{(-1)}(1)}\,|m(t)-m(s)| \\ &\le \|y(t)-y(s)\|_U + \frac{1}{U^{(-1)}(1)}\mathsf{E}\,|y(t)-y(s)| \\ &\le \|y(t)-y(s)\|_U + \frac{d_U}{U^{(-1)}(1)}\,\|y(t)-y(s)\|_U \\ &= \|y(t)-y(s)\|_U\left(1+\frac{d_U}{U^{(-1)}(1)}\right), \end{aligned} \tag{4.6}$$

where d_U is the constant from Lemma 2.10.

Therefore it follows from (4.5) that

$$\sup_{\rho(t,s)\le h} \|X(t)-X(s)\|_U \le \sigma_1(h).$$

Now the assertion of the Theorem 4.1 follows from the Theorem 2.15. □

Remark 4.2. It is easy to prove (as in 4.6) that

$$\|X(t_0)\|_U \le \left(1+\frac{d_U}{U^{(-1)}(1)}\right)\|y(t_0)\|\,. \tag{4.7}$$

Consider integrals $\int_{\mathcal{S}} f(t,s)p(s)d\mu(s) = \mathcal{I}(t)$. We suppose that all assumptions of Section 3 are true, but a function $f(t,s)$ depends on $t \in \mathbf{T}$, where $(\mathbf{T}, w)$ is a compact metric space. Suppose that $f(t,s)$ is continuous function of t.

Suppose that this integral exist. We can consider $f(t,s)$ as random processes on $\{\mathcal{S}, \mathcal{A}, m\}$ and $\mathcal{I}(t) = \int_{\mathcal{S}} f(t,s)p(s)d\mu(s) = \int_{\mathcal{S}} f(t,s)dm(s) = \mathsf{E}f(t)$.

Let $\xi_i(t)$, $i = 1,2,\dots,n$ be independent copies of $f(t,s)$. Let $Z_n(t) = \frac{1}{n}\sum_{i=1}^{n}\xi_i(t)$, $Z_n(t) \to \mathsf{E}f(t) = \mathcal{I}(t)$ with probability one for any $t \in \mathbf{T}$.

Definition 4.3. $Z_n(t)$ approach $\mathcal{I}(t)$ in the space $\mathbf{C(T)}$ with reliability $1-\delta > 0$ and accuracy $\varepsilon > 0$ if the following inequality holds:

$$P\{\sup_{t\in\mathbf{T}} |Z_n(t) - \mathcal{I}(t)| > \varepsilon\} \le \delta. \tag{4.8}$$

Theorem 4.4. *Let random process $f(t)$ belong to a space $L_U(\Omega)$, where $L_U(\Omega)$ has the property* **H** *with the constant C_U and the condition* g *hold for U. There*

exist a continuous increasing function $\sigma = \{\sigma(h), 0 \le h \le \sup_{t,s\in\mathbf{T}} w(t,s)\}$, *that*

$$\sup_{\rho(t,s)\le h} \left\| f(t) - f(s) \right\|_U \le \sigma(h) \tag{4.9}$$

and

$$\int_0^{\delta_0} U^{(-1)}(N_w(\sigma^{(-1)}(u)))du < \infty. \tag{4.10}$$

Then $Z_n(t)$ *approaches* $\mathcal{I}(t)$ *with reliability* $1-\delta$ *and accuracy* ε *in the space* $\mathbf{C(T)}$ *if the following inequality holds* $(U(\frac{\varepsilon\sqrt{n}}{\check{B}(t_0,\theta)}))^{-1} \le \delta$, *so*

$$n \ge \frac{\check{B}^2(\theta)U^{(-1)}\left(\frac{1}{\delta}\right)}{\varepsilon^2} \tag{4.11}$$

where $\check{B}(\theta) = (1 + \frac{d_U}{U^{(-1)}(1)})\|f(t)\|_U + \frac{1}{\theta(1-\theta)}\int_0^{\delta_0\theta} \nu(N_w(\sigma_1^{(-1)}(u))du$, $\sigma_1(h) = (1 + \frac{d_U}{U^{(-1)}(1)})\ \sigma(h)$, $\delta_0 = \sigma_1(\sup_{t,s\in\mathbf{T}} w(t,s))$, $0 < \theta < 1$, $\nu(n)$ *defined in Theorem* 2.14.

Proof. The function $f(t,s)$ is continuous. Therefore the process $f(t)$ is separable. Thus, it follows from (4.3) that

$$P\{\sup_{t\in\mathbf{T}} \sqrt{n}\,|Z_n(t) - m(t)| > \varepsilon\} \le \frac{1}{U\left(\frac{\varepsilon}{\check{B}(\theta)}\right)}.$$

Therefore

$$\begin{aligned} &P\{\sup_{t\in\mathbf{T}} |Z_n(t) - m(t)| > \varepsilon\} \\ &\quad = P\{\sup_{t\in\mathbf{T}} \sqrt{n}\,|Z_n(t) - m(t)| > \sqrt{n}\varepsilon\} \le \frac{1}{U\left(\frac{\varepsilon\sqrt{n}}{\check{B}(\theta)}\right)}. \end{aligned}$$

□

Example 4.5. Let $I(t) = \sqrt{2\pi}a\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{+\infty} f(x)\exp\{\frac{-x^2}{2a^2} - tx\}dx$, where $a > 0$, $|f(x)| < 1$ and $0 \le t \le T$. We keep the same notations as in Example 3.7. The estimation for $I(t)$ will be $I_n(t) = \sqrt{2\pi}a J_n$. Let put $U(x) = |x|^p$, $p \ge 2$, then according to the Theorem 4.4 we have that

$$\check{B}(\theta) = 2\|\eta(t)\|_p + \frac{1}{\theta(1-\theta)}\int_0^{\delta_0\theta} \nu(N_w(\sigma_1^{(-1)}(u))du,$$

where $\|\eta(t)\|_p = \exp\{\frac{pt^2a^2}{2}\} \le \exp\{\frac{pT^2a^2}{2}\}$.

Now estimate

$$\begin{aligned}\int_0^{\delta_0\theta} \nu(N_w(\sigma_1^{(-1)}(u))du &\le \int_0^{\delta_0\theta} \left(\frac{T}{2\sigma_1^{(-1)}(u)}+1\right)^{1/p} du \\ &\le \int_0^{\delta_0\theta} \left(\frac{3}{2}T\right)^{1/p} \left(\frac{1}{\sigma_1^{(-1)}(u)}\right)^{1/p} du \\ &= 2\left(\frac{3}{2}T\right)^{1/p} \int_0^{\delta_0\theta/2} \left(\frac{1}{\sigma^{(-1)}(v)}\right)^{1/p} dv.\end{aligned}$$

Find $\sigma(v)$, where $0 \le v \le T$, from the inequality (4.9).

$$\begin{aligned}\|\exp\{-\xi t\} - \exp\{-\xi s\}\|_p^p &= \mathsf{E}|\exp\{-\xi t\} - \exp\{-\xi s\}|^p \\ &= \mathsf{E}I\{\xi \ge 0\}\,|\exp\{-\xi t\} - \exp\{-\xi s\}|^p \\ &\quad + \mathsf{E}I\{\xi < 0\}\,|\exp\{-\xi t\} - \exp\{-\xi s\}|^p \\ &= \Delta_+ + \Delta_-.\end{aligned}$$

Let $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\beta > 1$ and $s > t$. Then

$$\begin{aligned}\Delta_- &= \mathsf{E}I\{\xi < 0\}\,|\exp\{-\xi t\} - \exp\{-\xi s\}|^p \\ &= \mathsf{E}I\{\xi < 0\}\,|\exp\{-\xi t\}(1 - \exp\{-\xi (s-t)\})|^p \\ &\le \mathsf{E}I\{\xi < 0\}\,|\exp\{-\xi t\}\,|\xi|\,(s-t)|^p \\ &\le (\mathsf{E}I\{\xi < 0\}\exp\{-\xi p t\beta\})^{1/\beta}\,(\mathsf{E}I\{\xi < 0\}\,|\xi|^{p\alpha})^{1/\alpha}(s-t)^p.\end{aligned}$$

Then $\Delta_- \le |t-s|^p \exp\{\frac{a^2p^2t^2\beta}{2}\}(\mathsf{E}|\xi|^{p\alpha})^{1/\alpha}$. Find similar Δ_+. Now $\sigma(v) = |t-s|C_p$, where $C_p = 2^{1/p}\exp\{\frac{a^2p\beta T^2}{2}\}(\mathsf{E}|\xi|^{p\alpha})^{1/p\alpha}$. Let's estimate

$$\begin{aligned}\mathsf{E}\,|\xi|^{p\alpha} &= \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{+\infty} |x|^{p\alpha}\exp\left\{\frac{-x^2}{2a^2}\right\}dx \\ &= \frac{a^{p\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} |t|^{p\alpha}\exp\left\{\frac{-t^2}{2}\right\}dt.\end{aligned}$$

Since $x^s \le (\frac{s}{e})\exp\{x\}$, then

$$\begin{aligned}\mathsf{E}\,|\xi|^{p\alpha} &\le \frac{a^{p\alpha}}{\sqrt{2\pi}}\left(\frac{p\alpha}{e}\right)^{p\alpha}\int_{-\infty}^{+\infty} \exp\{|t|\}\exp\left\{\frac{-t^2}{2}\right\}dt \\ &\le 2\exp\{1/2\}\,a^{p\alpha}\left(\frac{p\alpha}{e}\right)^{p\alpha}.\end{aligned}$$

We receive that $C_p = (2\exp\{1/2\})^{1/p\alpha} a^{\frac{p\alpha}{e}} 2^{1/p} \exp\{\frac{a^2 p\beta T^2}{2}\}$. Let substitute the obtained values, that $(\sigma_1(h) = 2\sigma(h))$

$$\check{B}(\theta) = 2\exp\left\{\frac{a^2 pT^2}{2}\right\} + \frac{1}{\theta(1-\theta)}\int_0^{\delta_0\theta/2}\left(\frac{C_p}{v}\right)^{1/p} dv$$
$$= 2\exp\left\{\frac{a^2 pT^2}{2}\right\} + \frac{C_p^{1/p}p}{\theta(1-\theta)(p-1)}\left(\frac{\delta_o\theta}{2}\right)^{1-1/p}.$$

Since $\delta_0 = \sigma_1(\sup_{t,s\in\mathbf{T}} w(t,s))$ and $\sup_{t,s\in\mathbf{T}} w(t,s) = T$ then $\delta_0 = \sigma_1(T) = 2\sigma(T) = 2TC_p$ and $\check{B}(\theta) = 2\exp\{\frac{a^2pT^2}{2}\} + \frac{2^{1/p-1}C_p pT}{(1-\theta)(p-1)}(T\theta)^{-1/p}$. According to equation (4.11) integral will be computed with accuracy ε and reliability δ, when the following inequality holds:

$$n \geq \inf_{p\geq 2, 0<\theta<1}\left(\frac{a^2 2\pi \check{B}^2(\theta)}{\varepsilon^2\delta^{2/p}}\right).$$

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