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THE BANACH SPACES $\mathbf{F}_\psi(\Omega)$ OF RANDOM VARIABLES

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Abstract. Some properties of random variables and stochastic processes belonging to the spaces $\mathbf{F}_\psi(\Omega)$ are studied.

1. Introduction

The Monte-Carlo method for the evaluation of the integrals

$$I(t) = \int \dots \int_{\mathbf{R}^d} f(t, \vec{x}) p(\vec{x})\, dx, \qquad t \in \mathbf{T},$$

is considered in the paper [1] (also see [2]), where $p(\vec{x})$ is the probability density of a certain random vector. The integrals considered in [1] depend on a parameter t (the particular case where the integrals do not depend on a parameter t is also considered in [1]). Sufficient conditions are found in [1] to evaluate the integrals with a given accuracy and reliability. The proofs of the results in [1] are based on the methods of the theory of stochastic processes in Orlicz spaces.

It became evident that the methods used in [1] are more efficient if one uses the so-called moment norms (more precisely, the Luxemburg norms) which are equivalent to the usual norms in Orlicz spaces. One can also use the theory of the spaces $\mathbf{F}_\psi(\Omega)$ to evaluate the integrals with a given accuracy and reliability. The norms in the spaces $\mathbf{F}_\psi(\Omega)$ are given by

$$\|\xi\|_\psi = \sup_{u \ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)},$$

where $\psi(u) > 0$ is a certain increasing function. The spaces $\mathbf{F}_\psi(\Omega)$ are introduced in the paper [3].

The current paper is devoted to a study of some properties of the spaces $\mathbf{F}_\psi(\Omega)$ that can be used for the Monte-Carlo method to evaluate the integrals with a given accuracy and reliability. However, the study of the spaces $\mathbf{F}_\psi(\Omega)$ has its own interest in view of several other applications, namely for constructing models of stochastic processes, their approximations, etc.

In a forthcoming paper, we plan to apply the results obtained below for a study of accuracy and reliability of Monte-Carlo methods.

The paper is organized as follows. Section 2 is devoted to the main properties of the spaces $\mathbf{F}_\psi(\Omega)$. A relationship between some spaces $\mathbf{F}_\psi(\Omega)$ and Orlicz spaces is considered in Section 3. Some bounds for the distributions of supremums are obtained in Section 4 for stochastic processes belonging to the spaces $\mathbf{F}_\psi(\Omega)$.

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2. $\mathbf{F}_\psi(\Omega)$-SPACES

Definition 2.1. Let $\psi(u) > 0$, $u \ge 1$, be an increasing continuous function such that $\psi(u) \to \infty$ as $u \to \infty$. We say that a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ if

$$\sup_{u \ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} < \infty.$$

It is proved in [3] that $\mathbf{F}_\psi(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_\psi = \sup_{u \ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)}. \tag{1}$$

Remark 2.1. It is obvious that $\mathbf{F}_\psi(\Omega)$ is a normed linear space. The proof that $\mathbf{F}_\psi(\Omega)$ is a Banach space is similar to the proof of a similar result that every Orlicz space of random variables is a Banach space (see [4]).

Theorem 2.1. *Let a random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$. Then*

$$\mathsf{P}\left\{|\xi| > x\right\} \le \inf_{u \ge 1} \frac{\|\xi\|_\psi^u\,(\psi(u))^u}{x^u} \tag{2}$$

for all $x > 0$.

Proof. The Chebyshev inequality implies that

$$\mathsf{P}\left\{|\xi| > x\right\} \le \frac{\mathsf{E}\,|\xi|^u}{x^u} \le \frac{\mathsf{E}\,|\xi|^u\,(\psi(u))^u}{(\psi(u))^u x^u} = \frac{\|\xi\|_\psi^u\,(\psi(u))^u}{x^u}$$

for all $u > 0$. □

Example 2.1. Let $\psi(u) = u^\alpha$ and $\alpha > 0$. We show that

$$\mathsf{P}\left\{|\xi| > x\right\} \le \exp\left\{-\frac{\alpha}{e}\left(\frac{x}{\|\xi\|_\psi}\right)^{1/\alpha}\right\}$$

for $x \ge e^\alpha \|\xi\|_\psi$. Indeed, let $b = \|\xi\|_\psi / x$. The minimum of the expression $b^u u^{\alpha u}$ is attained at the point $u = e^{-1} b^{-1/\alpha}$. Substituting this number in inequality (2), we prove the above bound.

Example 2.2. Similarly to Example 2.1, one can show that if $\psi(u) = e^{au}$ and $a > 0$, then

$$\mathsf{P}\left\{|\xi| > x\right\} \le \exp\left\{-\frac{\left(\ln \frac{x}{\|\xi\|_\psi}\right)^2}{4a}\right\}$$

for all $x \ge e^{2a} \|\xi\|_\psi$.

Example 2.3. Let $\psi(u) = e^{u^2}$. An optimization procedure similar to that in Examples 2.1 and 2.2 proves that

$$\mathsf{P}\left\{|\xi| > x\right\} \le \exp\left\{-\frac{2\left(\ln \frac{x}{\|\xi\|_\psi}\right)^{3/2}}{3^{3/2}}\right\}$$

for all $x \ge e^3 \|\xi\|_\psi$.

Definition 2.2. We say that $\mathbf{F}_\psi(\Omega)$ is a $\check{\mathbf{F}}_\psi(\Omega)$-space if the function $\psi(u)$ is such that

$$\sup_{u\ge 1}\frac{\psi(u+v)}{\psi(u)}<\infty \tag{3}$$

for all $v>0$.

Condition (3) obviously holds for the functions

1) $\psi(u)=e^{au^\beta}$ if $0<\beta\le 1$ and $a>0$,
2) $\psi(u)=Au^\alpha$ if $\alpha>0$ and $A>0$.

Definition 2.3 ([5, 4])**.** We say that a positive non-decreasing sequence $(\varkappa(n), n\ge 1)$ is an M-characteristic (a majorant) of a space $\mathbf{F}_\psi(\Omega)$ if

$$\left\|\max_{1\le i\le n}|\xi_i|\right\|_\psi \le \varkappa(n)\max_{1\le i\le n}\|\xi_i\|_\psi \tag{4}$$

for all random variables ξ_i, $i=1,2,\dots,n$, belonging to the space $\mathbf{F}_\psi(\Omega)$.

Theorem 2.2. *The sequence*

$$\varkappa(n)=\sup_{u\ge 1}\inf_{v>0} n^{\frac{1}{u+v}}\frac{\psi(u+v)}{\psi(u)} \tag{5}$$

is an M-characteristic (*a majorant*) *of the space $\mathbf{F}_\psi(\Omega)$.*

Proof. Let $\xi_1,\xi_2,\dots,\xi_n$ be a sequence of random variables belonging to the space $\mathbf{F}_\psi(\Omega)$. Then

$$\begin{aligned}\left\|\max_{1\le i\le n}|\xi_i|\right\|_\psi &= \sup_{u\ge 1}\frac{\left(\mathsf{E}\left(\max_{1\le i\le n}|\xi_i|\right)^u\right)^{1/u}}{\psi(u)} \le \sup_{u\ge 1}\frac{\left(\mathsf{E}\left(\max_{1\le i\le n}|\xi_i|\right)^{u+v}\right)^{\frac{1}{u+v}}}{\psi(u)}\\ &\le \sup_{u\ge 1}\max_{1\le i\le n} n^{\frac{1}{u+v}}\frac{\left(\mathsf{E}|\xi|^{u+v}\right)^{\frac{1}{u+v}}}{\psi(u+v)}\cdot\frac{\psi(u+v)}{\psi(u)}\\ &\le \max_{1\le i\le n}\|\xi_i\|_\psi \sup_{u\ge 1} n^{\frac{1}{u+v}}\frac{\psi(u+v)}{\psi(u)}\end{aligned}$$

for all $v>0$. Since $v>0$ is an arbitrary number, bound (4) implies Theorem 2.2. ☐

Example 2.4. Let $\psi(u)=e^{u^2}$. Then a majorant of the corresponding space $\mathbf{F}_\psi(\Omega)$ is equal to

$$\varkappa(n)=\exp\left\{\frac{3}{2^{2/3}}(\ln n)^{2/3}-1\right\}.$$

The minimum of the expression $n^{\frac{1}{u+v}}e^{v^2+2uv}$ is attained at the point $v=(\frac12\ln n)^{1/3}-u$. Substituting this number in equality (5), we complete the proof.

The evaluation of a majorant $\varkappa(n)$ is simpler for the spaces $\check{\mathbf{F}}_\psi(\Omega)$.

Corollary 2.1. *A sequence*

$$\varkappa(n)=\inf_{v>0} z(v)n^{\frac{1}{v+1}} \tag{6}$$

is a majorant of the space $\check{\mathbf{F}}_\psi(\Omega)$, where

$$z(v)=\sup_{u\ge 1}\frac{\psi(u+v)}{\psi(u)}.$$

Proof. Indeed, equality (5) implies that

$$\sup_{u\geq 1}\inf_{v>0} n^{\frac{1}{u+v}}\frac{\psi(u+v)}{\psi(u)} \leq \inf_{v>0}\sup_{u\geq 1} n^{\frac{1}{u+v}}\frac{\psi(u+v)}{\psi(u)} = \inf_{v>0} z(v) n^{\frac{1}{v+1}}. \qquad \square$$

Example 2.5. Let $\psi(u) = e^{au}$ and $a \neq 0$. Then a majorant is given by

$$\varkappa(n) = e^{2\sqrt{a\ln n}-a}.$$

Example 2.6. Let $\psi(u) = u^\alpha$ and $\alpha > 0$. Then a majorant is given by

$$\varkappa(n) = (\ln n)^\alpha \left(\frac{e}{\alpha}\right)^\alpha.$$

Note that one obtains the same majorants for the spaces considered in Examples 2.5 and 2.6 irrespective of which formula is used for the evaluation, either formula (5) or (6).

Definition 2.4. Let $\{S_k\}$ be an increasing sequence such that $S_k \geq 1$ and $S_k \to \infty$ as $k \to \infty$. Consider an increasing continuous function $\psi(u)$ such that $\psi(u) > 0$, $u \geq 1$, and $\psi(u) \to \infty$ as $u \to \infty$. Then we say that a random variable ξ belongs to the space $\mathbf{F}_{S_k,\psi,r}(\Omega)$ if

$$\sup_{k\geq r}\frac{\left(\mathsf{E}\,|\xi|^{S_k}\right)^{1/S_k}}{\psi(S_k)} < \infty.$$

Similarly to the preceding case, one can easily prove that the spaces $\mathbf{F}_{S_k,\psi,r}(\Omega)$ are Banach spaces with respect to the norms

$$\|\xi\|_{S_k,\psi,r} = \sup_{k\geq r}\frac{\left(\mathsf{E}\,|\xi|^{S_k}\right)^{1/S_k}}{\psi(S_k)}. \tag{7}$$

Theorem 2.3. *Let condition* (3) *hold for a function* ψ. *Assume that there exists a number* $C_r > 0$ *such that*

$$\frac{\psi(S_k)}{\psi(S_{k-1})} \leq C_r, \qquad k \geq r.$$

Then the spaces $\mathbf{F}_{S_k,\psi,r}(\Omega)$ *contain the same elements as the spaces* $\check{\mathbf{F}}_\psi(\Omega)$ *and the norms* (1) *and* (7) *are equivalent.*

Proof. Indeed,

$$\|\xi\|_{S_k,\psi,r} \leq \|\xi\|_\psi.$$

Now Lyapunov's inequality implies that

$$\begin{aligned}\frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} &\leq \frac{\left(\mathsf{E}\,|\xi|^{S_k}\right)^{1/S_k}}{\psi(u)} = \frac{\left(\mathsf{E}\,|\xi|^{S_k}\right)^{1/S_k}}{\psi(S_k)}\cdot\frac{\psi(S_k)}{\psi(u)} \\ &\leq \|\xi\|_{S_k,\psi,r}\cdot\frac{\psi(S_k)}{\psi(u)} \leq \|\xi\|_{S_k,\psi,r}\cdot\frac{\psi(S_k)}{\psi(S_{k-1})} \leq C_r\,\|\xi\|_{S_k,\psi,r}\end{aligned}$$

for $S_{k-1} \leq u \leq S_k$ and $k-1 \geq r$.

Further, for $1 \leq u \leq S_r$,

$$\frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} \leq \frac{\left(\mathsf{E}\,|\xi|^{S_r}\right)^{1/S_r}}{\psi(S_r)}\cdot\frac{\psi(S_r)}{\psi(u)} \leq C_2\,\|\xi\|_{S_k,\psi,r},$$

where

$$C_2 = \sup_{1\leq u\leq S_r}\frac{\psi(S_r)}{\psi(u)}.$$

Therefore

$$\|\xi\|_\psi \le \max(C_2, C_r)\, \|\xi\|_{S_k,\psi,r}\,. \qquad \square$$

Remark 2.2. Theorem 2.3 may fail if its main assumption does not hold.

The space corresponding to the sequence $S_k = 2k$ is the most important among the spaces $\mathbf{F}_{S_k,\psi,r}(\Omega)$ for our purposes. We denote by

$$\theta_{2,r}(\xi) = \sup_{k\ge r} \frac{\left(\mathsf{E}\,|\xi|^{2k}\right)^{1/2k}}{\psi(2k)}$$

the norm in this space. It is clear that if condition (3) holds for a function ψ, then the spaces $\check{\mathbf{F}}_\psi(\Omega)$ and $\mathbf{F}_{2k,\psi,r}(\Omega)$ coincide and the norms in these spaces are equivalent.

Indeed, according to Theorem 2.3,

$$\theta_{2,r}(\xi) \le \|\xi\|_\psi\,. \tag{8}$$

Note that

$$\sup_{k\ge r} \frac{\psi(2k)}{\psi(2k-2)} = \sup_{k\ge r} \frac{\psi(2k-2+2)}{\psi(2k-2)} \le \sup_{u\ge r} \frac{\psi(u+2)}{\psi(u)} = \overline{\psi}_r < \infty,$$

that is

$$\|\xi\|_\psi \le \widehat{\psi}_r \theta_{2,r}(\xi), \tag{9}$$

where

$$\widehat{\psi}_r = \max\left\{\overline{\psi}_r, C_2\right\}. \tag{10}$$

The proof of the following result is similar to that of Theorem 2.1.

Theorem 2.4. *Let a random variable ξ belong to the space $\mathbf{F}_{S_k,\psi,r}(\Omega)$. Then*

$$\mathsf{P}\{|\xi| > x\} \le \inf_{k\ge r} \frac{\|\xi\|_{S_k,\psi,r}^{S_k} (\psi(S_k))^{S_k}}{x^{S_k}} \tag{11}$$

for all $x > 0$.

In particular, if $S_k = 2k$, then

$$\mathsf{P}\{|\xi| > x\} \le \inf_{k\ge r} \frac{\|\xi\|_{2k,\psi,r}^{2k} (\psi(2k))^{2k}}{x^{2k}}$$

according to Theorem 2.3.

Theorem 2.5. *The sequence*

$$\varkappa(n) = \sup_{k\ge r} \inf_{v>0} n^{\frac{1}{S_k+v}} \frac{\psi(S_k+v)}{\psi(S_k)} \tag{12}$$

is an M-characteristic (majorant) of the space $\mathbf{F}_{S_k,\psi,r}(\Omega)$.

Proof. Similarly to the proof of Theorem 2.2 consider a sequence of random variables ξ_1, $\xi_2, \ldots, \xi_n$ in the space $\mathbf{F}_{S_k,\psi,r}(\Omega)$. Then

$$\left\|\max_{1\le i\le n}|\xi_i|\right\|_{S_k,\psi,r} = \sup_{k\ge r}\frac{\left(\mathsf{E}\left(\max_{1\le i\le n}|\xi_i|\right)^{S_k}\right)^{1/S_k}}{\psi(S_k)} \le \sup_{k\ge r}\frac{\left(\mathsf{E}\left(\max_{1\le i\le n}|\xi_i|\right)^{S_k+v}\right)^{\frac{1}{S_k+v}}}{\psi(S_k)}$$
$$\le \sup_{k\ge r}\max_{1\le i\le n} n^{\frac{1}{S_k+v}}\frac{\left(\mathsf{E}|\xi|^{S_k+v}\right)^{\frac{1}{S_k+v}}}{\psi(S_k+v)}\cdot\frac{\psi(S_k+v)}{\psi(S_k)}$$
$$\le \max_{1\le i\le n}\|\xi_i\|_{S_k,\psi,r}\sup_{k\ge r} n^{\frac{1}{S_k+v}}\frac{\psi(S_k+v)}{\psi(S_k)}$$

for all $v>0$. The latter inequality and bound (4) complete the proof of Theorem 2.5. □

Below we list some examples of random variables belonging to the spaces $\mathbf{F}_\psi(\Omega)$.

Example 2.7. If a random variable ξ is bounded, that is, if $|\xi| < \text{const}$ with probability one, then ξ belongs to all of the spaces $\mathbf{F}_\psi(\Omega)$.

Example 2.8. Every random variable with the Laplace distribution (the density of the Laplace distribution is $p(x)=\frac{1}{2}e^{-|x|}$) belongs to the space $\mathbf{F}_\psi(\Omega)$ for $\psi(u)=u$. Indeed,

$$\sqrt[k]{\mathsf{E}|\xi|^k} = \sqrt[k]{k!}\sim k.$$

Example 2.9. Every normal random variable $\xi = N(0,1)$ belongs to the space $\mathbf{F}_\psi(\Omega)$ if $\psi(u)=u^{1/2}$. Indeed,

$$\sqrt[2l]{\mathsf{E}|\xi|^{2l}} = \sqrt[2l]{\frac{(2l)!}{2^l l!}}\sim l^{1/2}.$$

Definition 2.5 ([1])**.** We say that a Banach space $B(\Omega)$ of random variables possesses property **H** if there exists an absolute constant C_B such that

$$\left\|\sum_{i=1}^n \xi_i\right\|^2 \le C_B\sum_{i=1}^n\|\xi_i\|^2 \tag{13}$$

for all $n\ge 1$ and all independent centered random variables $\xi_1,\xi_2,\ldots,\xi_n$ belonging to the space $B(\Omega)$.

Our aim is to find conditions for the processes $\check{\mathbf{F}}_\psi(\Omega)$ to possess property **H** and to evaluate the corresponding constant C_B.

Theorem 2.6. *Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent centered random variables belonging to the space $\check{\mathbf{F}}_\psi(\Omega)$. Assume that the ξ_k are symmetric random variables for $k\ge\max(r,2)$ and that*

$$C_{2k}^{2l}\frac{(\psi(2l))^{2l}(\psi(2k-2l))^{2k-2l}}{(\psi(2k))^{2k}}\le C_k^l,\qquad l=1,\ldots,k-1. \tag{14}$$

Then

$$\theta_{2,r}^2\left(\sum_{i=1}^n\xi_i\right)\le\sum_{i=1}^n\theta_{2,r}^2(\xi_i). \tag{15}$$

If the random variables $\xi_1,\xi_2,\ldots,\xi_n$ are not necessarily symmetric, then condition (14) *implies that*

$$\theta_{2,r}^2\left(\sum_{i=1}^n\xi_i\right)\le 4\sum_{i=1}^n\theta_{2,r}^2(\xi_i). \tag{16}$$

If $\xi_1, \xi_2, \dots, \xi_n$ are not symmetric random variables and

$$C_{2k}^{2l}\left(1+\frac{k}{3}\right)\frac{(\psi(2l))^{2l}\,(\psi(2k-2l))^{2k-2l}}{(\psi(2k))^{2k}} \le C_k^l, \qquad l=1,\dots,k-1, \tag{17}$$

then

$$\theta_{2,r}^2\left(\sum_{i=1}^n \xi_i\right) \le \sum_{i=1}^n \theta_{2,r}^2\left(\xi_i\right). \tag{18}$$

We need some auxiliary results to prove Theorem 2.6.

Lemma 2.1. *Let ξ and η be random variables belonging to a space $\check{\mathbf{F}}_\psi(\Omega)$. If ξ and η are independent and $\mathsf{E}\,\eta = 0$, then*

$$\|\xi\|_\psi \le \|\xi-\eta\|_\psi. \tag{19}$$

Proof. The Fubini theorem implies that

$$\mathsf{E}\,|\xi-\eta|^u = \mathsf{E}_\xi\left(\mathsf{E}_\eta\,|\xi-\eta|^u\right) \tag{20}$$

for $u > 1$, where the symbol E_ξ stands for the conditional mathematical expectation with respect to ξ. Similarly, E_η denotes the conditional mathematical expectation with respect to η. The Lyapunov inequality implies that

$$\mathsf{E}_\xi\,|\xi-\eta|^u \ge \left(\mathsf{E}_\xi\,|\xi-\eta|\right)^u \ge \left|\mathsf{E}_\xi\left(\xi-\eta\right)\right|^u = \left|\xi-\mathsf{E}_\xi\,\eta\right|^u = |\xi|^u$$

for $u \ge 1$.

Hence (20) implies that

$$\mathsf{E}\,|\xi-\eta|^u \ge \mathsf{E}\,|\xi|^u.$$

Relation (19) obviously follows from the latter inequality. □

Proof of Theorem 2.6. Let $\xi_1, \xi_2, \dots, \xi_n$ be symmetric random variables. Then all odd moments are equal to zero. Thus

$$\mathsf{E}\left(\xi_1+\xi_2\right)^{2k} = \mathsf{E}\,\xi_1^{2k} + \sum_{s=2}^{2k-2} C_{2k}^s\,\mathsf{E}\,\xi_1^s\xi_2^{2k-s} + \mathsf{E}\,\xi_2^{2k} = \mathsf{E}\,\xi_1^{2k} + \sum_{r=1}^{k-1} C_{2k}^{2r}\,\mathsf{E}\,\xi_1^{2r}\,\mathsf{E}\,\xi_2^{2k-2r} + \mathsf{E}\,\xi_2^{2k}.$$

Since $\mathsf{E}\,|\xi_i|^{2k} \le (\psi(2k))^{2k}\,\theta_{2,r}^{2k}\left(\xi_i\right)$, we get

$$\begin{aligned}
\frac{\mathsf{E}\left(\xi_1+\xi_2\right)^{2k}}{(\psi(2k))^{2k}} &\le \theta_{2,r}^{2k}\left(\xi_1\right) + \sum_{r=1}^{k-1} C_{2k}^{2r}\,(\psi(2r))^{2r}\,(\psi(2k-2r))^{2k-2r}\,\frac{\theta_{2,r}^{2r}\left(\xi_1\right)\theta_{2,r}^{2k-2r}\left(\xi_2\right)}{(\psi(2k))^{2k}} \\
&\quad + \theta_{2,r}^{2k}\left(\xi_2\right) \\
&\le \theta_{2,r}^{2k}\left(\xi_1\right) + \sum_{r=1}^{k-1} C_k^r\theta_{2,r}^{2r}\left(\xi_1\right)\theta_{2,r}^{2k-2r}\left(\xi_2\right) + \theta_{2,r}^{2k}\left(\xi_2\right) \\
&= \left(\theta_{2,r}^2\left(\xi_1\right)+\theta_{2,r}^2\left(\xi_2\right)\right)^k.
\end{aligned}$$

The latter inequality implies (15) for $n = 2$.

Now let $\xi_1, \xi_2, \dots, \xi_n$ be independent centered random variables belonging to the space $\check{\mathbf{F}}_\psi(\Omega)$ and let $\xi_1^*, \xi_2^*, \dots, \xi_n^*$ be independent of ξ_i, $i = 1, \dots, n$, jointly independent random variables such that ξ_i^* has the same distribution as ξ_i. Then the random variables $\xi_i - \xi_i^*$ are symmetric. Lemma 2.1 implies that

$$\theta_{2,r}^2\left(\sum_{i=1}^n \xi_i\right) \le \theta_{2,r}^2\left(\sum_{i=1}^n (\xi_i - \xi_i^*)\right) \le \sum_{i=1}^n \theta_{2,r}^2\left(\xi_i - \xi_i^*\right)$$
$$\le \sum_{i=1}^n \left(\theta_{2,r}(\xi_i) + \theta_{2,r}(\xi_i^*)\right)^2 = 4\sum_{i=1}^n \theta_{2,r}^2\left(\xi_i\right),$$

where $\theta_{2,r}\left(\xi_i\right) = \theta_{2,r}\left(\xi_i^*\right)$. The proof of inequality (16) is complete.

The proof of inequality (18) is similar to that of (15). Since

$$\mathsf{E}\left(\xi_1 + \xi_2\right)^{2k} = \mathsf{E}\,\xi_1^{2k} + \sum_{s=2}^{2k-2} C_{2k}^s\,\mathsf{E}\,\xi_1^s \xi_2^{2k-s} + \mathsf{E}\,\xi_2^{2k}$$

and

$$\left|\mathsf{E}\,\xi_1^k \xi_2^{2k-s}\right| \le \frac{1}{2}\left(\mathsf{E}\left|\xi_1\right|^{s+1}\mathsf{E}\left|\xi_2\right|^{2k-s-1} + \mathsf{E}\left|\xi_1\right|^{s-1}\mathsf{E}\left|\xi_2\right|^{2k-s+1}\right)$$

for even s, we have

$$\mathsf{E}\left(\xi_1 + \xi_2\right)^{2k} \le \mathsf{E}\left|\xi_1\right|^{2k} + \sum_{l=1}^{k-1} R_{2k}^{2l}\,\mathsf{E}\left|\xi_1\right|^{2l}\mathsf{E}\left|\xi_2\right|^{2k-2l} + \mathsf{E}\left|\xi_2\right|^{2k},$$

where

$$R_{2k}^2 = R_{2k}^{2k-2} = C_{2k}^2 + \frac{1}{2}C_{2k}^3, \qquad R_{2k}^{2l} = C_{2k}^{2l} + \frac{1}{2}\left(C_{2k}^{2l+1} + C_{2k}^{2l-1}\right), \qquad l \ne 1, \quad l \ne k-1.$$

It is easy to prove that $R_{2k}^{2l} \le \left(1 + \frac{k}{3}\right)C_{2k}^{2l}$. Therefore

$$\begin{aligned}
C_{2k}\,\mathsf{E}\left(\xi_1 + \xi_2\right)^{2k} \le{}& \frac{\mathsf{E}\left|\xi_1\right|^{2k}}{(\psi(2k))^{2k}} \\
&+ \sum_{l=1}^{k-1} C_{2k}^{2l}\left(1 + \frac{k}{3}\right)\frac{\left(\psi(2l)\right)^{2l}\left(\psi(2k-2l)\right)^{2k-2l}}{\left(\psi(2k)\right)^{2k}}\left(\frac{\mathsf{E}\left|\xi_1\right|^{2l}}{(\psi(2l))^{2l}}\right) \\
&\times\left(\frac{\mathsf{E}\left|\xi_2\right|^{2k-2l}}{(\psi(2k-2l))^{2k-2l}}\right) \\
&+ \frac{\mathsf{E}\left|\xi_2\right|^{2k}}{(\psi(2k))^{2k}} \\
\le{}& \theta_{2,r}^{2k}\left(\xi_1\right) + \sum_{l=1}^{k-1} C_k^l \theta_{2,r}^{2l}\left(\xi_1\right)\theta_{2,r}^{2k-2l}\left(\xi_2\right) + \theta_{2,r}^{2k}\left(\xi_2\right) \\
={}& \left(\theta_{2,r}^2\left(\xi_1\right) + \theta_{2,r}^2\left(\xi_2\right)\right)^k.
\end{aligned}$$

The latter inequality implies (15) for $n = 2$ and thus Theorem 2.6 is proved. □

Theorem 2.6 and inequalities (8) and (9) yield the following result.

Corollary 2.2. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent centered random variables belonging to a space $\check{\mathbf{F}}_\psi(\Omega)$. Assume that the ξ_k are symmetric random variables for $k \ge \max(r, 2)$ and that*

$$C_{2k}^{2l}\frac{\left(\psi(2l)\right)^{2l}\left(\psi(2k-2l)\right)^{2k-2l}}{\left(\psi(2k)\right)^{2k}} \le C_k^l, \qquad l = 1, \ldots, k-1.$$

Then

$$\left\|\sum_{i=1}^n \xi_i\right\|_\psi^2 \le \widehat{\psi}_r^2 \sum_{i=1}^n \|\xi_i\|_\psi^2,$$

where $\widehat{\psi}_r^2$ is defined by (10).

If the random variables ξ_i are not necessarily symmetric, then condition (14) *implies that*

$$\left\|\sum_{i=1}^n \xi_i\right\|_\psi^2 \le 4\widehat{\psi}_r^2 \sum_{i=1}^n \|\xi_i\|_\psi^2.$$

If the ξ_i are not necessarily symmetric random variables and

$$C_{2k}^{2l}\left(1+\frac{k}{3}\right)\frac{(\psi(2l))^{2l}\,(\psi(2k-2l))^{2k-2l}}{(\psi(2k))^{2k}} \le C_k^l, \qquad l=1,\dots,k-1,$$

then

$$\left\|\sum_{i=1}^n \xi_i\right\|_\psi^2 \le \widehat{\psi}_r^2 \sum_{i=1}^n \|\xi_i\|_\psi^2.$$

Below are some examples of the spaces $\mathbf{F}_\psi(\Omega)$ for which the above results hold.

Lemma 2.2. *The inequality*

$$C_{2k}^{2l} \le C_k^l \frac{k^k}{l^l(k-l)^{k-l}}\frac{1}{\sqrt{2}}\exp\left\{\frac{1}{8}\left(\frac{1}{k}+\frac{1}{k-1}+1\right)\right\}$$

holds for $k\ge 2$ and $1\le l\le k-1$.

Proof. Since

$$C_{2k}^{2l} = C_k^l \frac{C_{2k}^{2l}}{C_k^l}$$

and $n! = \sqrt{2\pi n}n^n e^{-n}e^{\theta_n}$ by Stirling's formula, where $|\theta_n| < \frac{1}{12n}$, we have

$$\begin{aligned}\frac{C_{2k}^{2l}}{C_k^l} &= \frac{(2k)!\,l!(k-l)!}{(2l)!(2k-2l)!\,k!} = \frac{k^{2k}l^l(k-l)^{k-l}}{\sqrt{2}l^{2l}(k-l)^{2(k-l)}k^k}\exp\{\theta_{2k}+\theta_{2l}+\theta_k+\theta_l+\theta_{k-l}\}\\
&\le \frac{k^k}{l^l(k-l)^{k-l}}\frac{1}{\sqrt{2}}\exp\left\{\frac{1}{24k}+\frac{1}{24l}+\frac{1}{24(k-l)}+\frac{1}{12k}+\frac{1}{12l}+\frac{1}{12(k-l)}\right\}\\
&\le \frac{k^k}{l^l(k-l)^{k-l}}\frac{1}{\sqrt{2}}\exp\left\{\frac{1}{8}\left(\frac{1}{k}+\frac{1}{k-1}+1\right)\right\}.\end{aligned}$$ □

Example 2.10. Consider the space $\mathbf{F}_\psi(\Omega)$ for $\psi(u)=u^\alpha$ and $\alpha\ge\frac{1}{2}$. We show that property **H** holds in this case. Indeed,

$$\begin{aligned}C_{2k}^{2l}\frac{(2l)^{2l\alpha}(2k-2l)^{(2k-2l)\alpha}}{(2k)^{2k\alpha}} &= C_{2k}^{2l}\left(\frac{l^{2l}(k-l)^{(2k-2l)}}{k^{2k}}\right)^\alpha\\
&\le C_k^l\frac{l^{(2\alpha-1)l}(k-l)^{(2\alpha-1)(k-l)}}{k^{(2\alpha-1)k}}\frac{1}{\sqrt{2}}\exp\left\{\frac{1}{8}\left(\frac{1}{k}+\frac{1}{k-1}+1\right)\right\}.\end{aligned}$$

It is obvious that inequality (14) holds for $\alpha\ge\frac{1}{2}$ and $k>2$, that is, the space $\mathbf{F}_\psi(\Omega)$ possesses the property **H** if $\psi(u)=u^\alpha$. One can also show that a space $\mathbf{F}_\psi(\Omega)$ does not possess property **H** if $\alpha<\frac{1}{2}$.

Example 2.11. Lemma 2.2 implies that the space $\mathbf{F}_\psi(\Omega)$ possesses property **H** if $\psi(u)=e^{au}$ and $a>0$.

3. A relationship between the spaces $\mathbf{F}_\psi(\Omega)$ and Orlicz spaces

Definition 3.1 ([4]). We say that an even continuous function $U = (U(x), x \in \mathbb{R})$ is a $\mathcal{C}$-function if $U(0) = 0$ and $U(x)$ is increasing for $x > 0$.

Definition 3.2 ([4]). Let U be an arbitrary $\mathcal{C}$-function. Consider the family of random variables such that, for every $\xi \in L_U(\Omega)$, there exists a constant $r_\xi > 0$ for which $\mathsf{E}\, U(\xi/r_\xi) < \infty$. This family is called the Orlicz space of random variables and is denoted by $L_U(\Omega)$.

An Orlicz space $L_U(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_U = \inf\left\{r > 0; \mathsf{E}\, U\left(\frac{\xi}{r}\right) \le 1\right\}.$$

This norm is called the Luxemburg norm.

Consider an Orlicz $\mathcal{C}$-function

$$U(x) = \begin{cases} \left(\frac{e\alpha}{2}\right)^{2/\alpha} x^2, & \text{if } |x| \le x_\alpha, \\ \exp\left\{|x|^\alpha\right\}, & \text{if } |x| > x_\alpha, \end{cases} \tag{21}$$

where $x_\alpha = (2/\alpha)^{1/\alpha}$, $0 < \alpha < 1$. Then $L_U(\Omega)$ is called the Orlicz space generated by the function $U(x)$.

Consider the function $U_1(x) = \exp\{|x|^\alpha\}$, $0 < \alpha \le 1$. Let $\mathcal{S}_{U_1}(\Omega)$ denote the family of random variables ξ for which there exists a number r such that $\mathsf{E}\, U_1\left(\frac{\xi}{r}\right) < \infty$. In the space $\mathcal{S}_{U_1}(\Omega)$, consider the functional

$$\langle\langle\xi\rangle\rangle_{U_1} = \inf\left\{r > 0; \mathsf{E}\, U_1\left(\frac{\xi}{r}\right) \le 2\right\}.$$

Lemma 3.1. *A random variable ξ belongs to an Orlicz space $L_U(\Omega)$ if and only if $\xi \in \mathcal{S}_{U_1}(\Omega)$ and*

$$\|\xi\|_U \le \left(e^{2/\alpha+2}\right) \langle\langle\xi\rangle\rangle_{U_1},$$
$$\langle\langle\xi\rangle\rangle_{U_1} \le \|\xi\|_U \left(e^{2/\alpha}+1\right)^{1/\alpha}.$$

Proof. Let $r > 0$. Then

$$\begin{aligned}\mathsf{E}\, U\left(\frac{\xi}{r}\right) &= \mathsf{E}\, U\left(\frac{\xi}{r}\right) \mathbb{I}\left\{\frac{|\xi|}{r} \le x_\alpha\right\} + \mathsf{E}\, U\left(\frac{\xi}{r}\right) \mathbb{I}\left\{\frac{|\xi|}{r} > x_\alpha\right\} \\ &\le U(x_\alpha) + \mathsf{E}\exp\left\{\left|\frac{\xi}{r}\right|^\alpha\right\} \\ &= e^{2/\alpha} + \mathsf{E}\exp\left\{\left|\frac{\xi}{r}\right|^\alpha\right\}.\end{aligned}$$

Now let $r = \langle\langle\xi\rangle\rangle_{U_1}$. Then $\mathsf{E}\, U(\xi/\langle\langle\xi\rangle\rangle_{U_1}) \le e^{2/\alpha} + 2$. Since $U(\alpha x) \le \alpha U(x)$ for $0 < \alpha < 1$ (see [4, Lemma 2.2.2]),

$$\mathsf{E}\, U\left(\frac{\xi}{\langle\langle\xi\rangle\rangle_{U_1}\left(e^{2\alpha}+2\right)}\right) \le \frac{1}{e^{2\alpha}+2}\, \mathsf{E}\, U\left(\frac{\xi}{\langle\langle\xi\rangle\rangle_{U_1}}\right) \le 1.$$

Hence

$$\|\xi\|_U \le \left(e^{2/\alpha+2}\right) \langle\langle\xi\rangle\rangle_{U_1}. \tag{22}$$

Now we prove the second inequality. It is easy to see that

$$\mathsf{E}\exp\left\{\left|\frac{\xi}{r}\right|^\alpha\right\} = \mathsf{E}\exp\left\{\left|\frac{\xi}{r}\right|^\alpha\right\}\mathbb{I}\left\{\frac{|\xi|}{r} < x_\alpha\right\} + \mathsf{E}\exp\left\{\left|\frac{\xi}{r}\right|^\alpha\right\}\mathbb{I}\left\{\frac{|\xi|}{r} \ge x_\alpha\right\}$$
$$\le \exp\left\{(x_\alpha)^\alpha\right\} + \mathsf{E}\,U\left(\frac{\xi}{r}\right).$$

Putting $r = \|\xi\|_U$, we obtain

$$\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U}\right|^\alpha\right\} \le e^{2/\alpha} + 1. \tag{23}$$

Using the inequality $\exp\{|ax|\} - 1 \le a\left(\exp\{|x|\} - 1\right)$ for $0 < a \le 1$, we get

$$\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U}\right|^\alpha \frac{1}{e^{2/\alpha}+1}\right\} - 1 \le \frac{1}{e^{2/\alpha}+1}\left(\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U}\right|^\alpha\right\} - 1\right).$$

Thus bound (23) implies that

$$\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U}\right|^\alpha \frac{1}{e^{2/\alpha}+1}\right\} - 1 \le \frac{1}{e^{2/\alpha}+1}\left(\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U}\right|^\alpha\right\} - 1\right)$$
$$\le \frac{1}{e^{2/\alpha}+1}\left(e^{2/\alpha}+1-1\right) = \frac{e^{2/\alpha}}{e^{2/\alpha}+1}.$$

As a result we obtain

$$\mathsf{E}\exp\left\{\left|\frac{\xi}{\|\xi\|_U\left(e^{2/\alpha}+1\right)^{1/\alpha}}\right|^\alpha\right\} \le \frac{e^{2/\alpha}}{e^{2/\alpha}+1} + 1 \le 2.$$

This implies that $\langle\langle\xi\rangle\rangle_{U_1} \le \|\xi\|_U\left(e^{2/\alpha}+1\right)^{1/\alpha}$. □

Lemma 3.2. *If* $0 < \alpha < 1$, *then*

$$\langle\langle\xi\rangle\rangle_{U_1} \ge \alpha^{1/\alpha}e^{1/\alpha}\left(\sup_{n\ge1}\frac{\left(\mathsf{E}|\xi|^n\right)^{1/n}}{n^{1/\alpha}}\right).$$

Proof. Since

$$x^n\exp\{-x^\alpha\} \le \left(\frac{n}{\alpha}\right)^{n/\alpha}\exp\left\{-\frac{n}{\alpha}\right\}$$

and

$$x^n \le \exp\{x^\alpha\}\left(\frac{n}{\alpha}\right)^{n/\alpha}\exp\left\{-\frac{n}{\alpha}\right\},$$

we get

$$\frac{\mathsf{E}|\xi|^n}{r^n} \le \mathsf{E}\exp\left\{\left(\frac{|\xi|}{r}\right)^\alpha\right\}\left(\frac{n}{\alpha}\right)^{n/\alpha}\exp\left\{-\frac{n}{\alpha}\right\},$$
$$\mathsf{E}|\xi|^n \le \langle\langle\xi\rangle\rangle_{U_1}^n 2\left(\frac{n}{\alpha}\right)^{n/\alpha}\exp\left\{-\frac{n}{\alpha}\right\},$$
$$\left(\mathsf{E}|\xi|^n\right)^{1/n} \le \langle\langle\xi\rangle\rangle_{U_1}2^{1/n}\left(\frac{n}{\alpha}\right)^{1/\alpha}\exp\left\{-\frac{1}{\alpha}\right\}.$$

In view of the inequality $\langle\langle\xi\rangle\rangle_{U_1} = \inf\left\{r>0; \mathsf{E}\exp\left\{(\xi/r)^\alpha\right\}\right\} \le 2$, we obtain

$$\langle\langle\xi\rangle\rangle_{U_1} \ge \left(\mathsf{E}|\xi|^n\right)^{1/n}\frac{1}{2^{1/n}\left(\frac{n}{\alpha}\right)^{1/\alpha}\exp\left\{-\frac{1}{\alpha}\right\}} \ge \frac{\left(\mathsf{E}|\xi|^n\right)^{1/n}}{n^{1/\alpha}}\alpha^{1/\alpha}e^{1/\alpha}.$$ □

Lemma 3.3. *If* $0 < \alpha < 1$, *then*

$$\langle\langle\xi\rangle\rangle_{U_1} \le \left(1 + \frac{e^{1/12}}{\sqrt{2\pi}}\right)^{1/\alpha}e^{1/\alpha}\left(\sup_{n\ge1}\frac{\left(\mathsf{E}|\xi|^n\right)^{1/n}}{n^{1/\alpha}}\right).$$

Proof. The Lyapunov inequality implies that

$$\mathsf{E}\,|\xi|^{n\alpha} \le \left(\mathsf{E}\,|\xi|^{n}\right)^{\alpha}$$

for $0<\alpha<1$.

Put $J_\alpha = \mathsf{E}\exp\left\{|\xi|^\alpha/r^\alpha\right\}-1$. Then

$$J_\alpha = \sum_{n=1}^{\infty}\frac{\mathsf{E}\,|\xi|^{n\alpha}}{n!\,r^{\alpha n}} \le \sum_{n=1}^{\infty}\frac{\left(\mathsf{E}\,|\xi|^{n}\right)^{\alpha}}{n!\,r^{\alpha n}}.$$

Let

$$\hat z = \sup_{n\ge 1}\frac{\left(\mathsf{E}\,|\xi|^{n}\right)^{1/n}}{n^{1/\alpha}}.$$

Since $\mathsf{E}\,|\xi|^{n} \le \hat z^{n} n^{n/\alpha}$, we conclude that

$$J_\alpha \le \sum_{n=1}^{\infty}\frac{\hat z^{n\alpha} n^{n}}{n!\,r^{\alpha n}}.$$

By Stirling's formula,

$$J_\alpha \le \sum_{n=1}^{\infty}\left(\frac{\hat z^\alpha e}{r^\alpha}\right)^{n}\frac{e^{1/12n}}{\sqrt{2\pi n}} \le \frac{e^{1/12}}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\frac{\hat z^\alpha e}{r^\alpha}\right)^{n}.$$

Let $r=\hat z e^{1/\alpha}/s^{1/\alpha}$, where $0<s<1$. Then

$$J_\alpha \le \frac{e^{1/12}}{\sqrt{2\pi}}\sum_{n=1}^{\infty}s^{n} = \frac{e^{1/12}}{\sqrt{2\pi}}\,\frac{s}{1-s}.$$

If $s=\left(1+e^{1/12}/\sqrt{2\pi}\right)^{-1}$, then

$$\mathsf{E}\exp\left\{\frac{|\xi|^\alpha}{\left(\frac{1}{s^{1/\alpha}}\hat z e^{1/\alpha}\right)^{\alpha}}\right\} \le 2.$$

This completes the proof of Lemma 3.3. □

Theorem 3.1. *Let a function $U(x)$ be defined by equality* (21). *Then the Orlicz space $L_U(\Omega)$ contains the same elements as the spaces $\mathbf{F}_\psi(\Omega)$ for $\psi(u)=u^{1/\alpha}$. Moreover, the norms in these spaces are equivalent.*

Theorem 3.1 follows from Lemmas 3.1, 3.2, and 3.3 and Theorem 2.3.

Other cases where the Orlicz spaces and $\mathbf{F}_\psi(\Omega)$ are equivalent are considered in the book [4] and in the paper [6].

4. Stochastic processes

Definition 4.1. Let $\mathbf{F}^*_\psi(\Omega)$ denote one of the following Banach spaces of random variables: either $\mathbf{F}_\psi(\Omega)$ or $\check{\mathbf{F}}_\psi(\Omega)$ or $\mathbf{F}_{S_k,\psi}(\Omega)$. The norm in the space $\mathbf{F}^*_\psi(\Omega)$ is denoted by $\|\cdot\|$.

Let $X=\{X(t),t\in\mathbf{T}\}$ be a stochastic process, let $\mathbf{T}=(\mathbf{T},\rho)$ be a compact metric space, and let ρ be a metric in $\mathbf{T}$. Let $N(u)$ denote the metric capacity of the space $(\mathbf{T},\rho)$. If $\gamma=\sigma\left(\sup_{t,s\in\mathbf{T}}\rho(t,s)\right)$, then we put $\varepsilon_k=\sigma^{(-1)}\left(\gamma p^k\right)$ for $k=0,1,2,\dots$ and $p\in(0,1)$.

Let $\mathbf{V}_{\varepsilon_k}$ be the set of centers of a minimal $\{\varepsilon_k\}$ net. We say that a set of closed balls is called a minimal $\{\varepsilon_k\}$ net if the radiuses do not exceed ε_k, if the balls cover $(\mathbf{T},\rho)$, that is, $\mathbf{V}=\bigcup_{k=0}^{\infty}\mathbf{V}_{\varepsilon_k}$, and if the set contains the minimal number of balls with the latter two properties.

If $k = 0$, then

$$\varepsilon_0 = \sigma^{(-1)}(\gamma) = \sigma^{(-1)}\left(\sigma\left(\sup_{t,s\in\mathbf{T}}\rho(t,s)\right)\right) = \sup_{t,s\in\mathbf{T}}\rho(t,s).$$

Definition 4.2. We say that a stochastic process X belongs to the space $\mathbf{F}_\psi^*(\Omega)$ if the random variable $X(t)$ belongs to the space $\mathbf{F}_\psi^*(\Omega)$ for all t.

Theorem 4.1. *Let $X(t)$ be a separable stochastic process in $(\mathbf{T},\rho)$ that belongs to the space $\mathbf{F}_\psi^*(\Omega)$. Assume that*

$$\sup_{\rho(t,s)\le h}\|X(t)-X(s)\| \le \sigma(h),$$

where $\sigma(h)$ is a continuous increasing function such that $\sigma(0)=0$. If

$$\int_0^\varepsilon \varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du < \infty$$

for all $\varepsilon > 0$ and if $\sup_{t\in\mathbf{T}}|X(t)| \in \mathbf{F}_\psi^*(\Omega)$, then

$$\left\|\sup_{t\in\mathbf{T}}|X(t)|\right\| \le B(p),$$

where

$$B(p) = \inf_{t\in\mathbf{T}}\|X(t)\| + \frac{1}{p(1-p)}\int_0^{\gamma p}\varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du$$

and where $\varkappa(n)$ is a majorant of the space $\mathbf{F}_\psi^*(\Omega)$.

Proof. Given an arbitrary $u > 1$, we get

$$\begin{aligned}\mathsf{P}\{|X(t)-X(s)|>\varepsilon\} &\le \frac{\mathsf{E}|X(t)-X(s)|^u}{\varepsilon^u} \le \frac{\|X(t)-X(s)\|_{\mathbf{F}_\psi}^u(\psi(u))^u}{\varepsilon^u}\\ &\le \frac{\sigma(\rho(t,s))(\psi(u))^u}{\varepsilon^u}.\end{aligned}$$

Hence

$$\mathsf{P}\{|X(t)-X(s)|>\varepsilon\}\to 0 \quad \text{as } \rho(t,s)\to 0$$

for all $\varepsilon > 0$. Thus the stochastic process X is continuous in probability, whence we conclude that $\mathbf{V}$ is a set of separability, that is,

$$\sup_{t\in\mathbf{T}}|X(t)| = \sup_{t\in\mathbf{V}}|X(t)|.$$

Consider the mapping $\alpha_k(t)$, $t\in\mathbf{V}$, such that $\alpha_k(t)$ is a point of $\mathbf{V}_{\varepsilon_k}$ for which $\rho(t,\alpha_k(t))\le\varepsilon_k$ (if $t\in\mathbf{V}_{\varepsilon_k}$, then $\alpha_k(t)=t$).

Let t be an arbitrary point of $\mathbf{V}$. If $t\in\mathbf{V}_{\varepsilon_m}$ for some integer number m, then we put $t_m = t$ and $t_{m-1}=\alpha_{m-1}(t_m)$, $t_{m-2}=\alpha_{m-2}(t_{m-1})$, ..., $t_1=\alpha_1(t_2)$, $t_0=\alpha_0(t_1)$, and accordingly

$$\begin{aligned}X(t) &= X(t_m)\\ &= X(t_m)-X(t_{m-1})+X(t_{m-1})-X(t_{m-2})+X(t_{m-2})-\dots\\ &\quad -X(t_1)-X(t_0)+X(t_0),\end{aligned}$$

$$\begin{aligned}|X(t)| &\le |X(t_m)-X(t_{m-1})|+|X(t_{m-1})-X(t_{m-2})|+\dots+|X(t_1)-X(t_0)|+|X(t_0)|\\ &\le \max_{t\in\mathbf{V}_{\varepsilon_m}}|X(t)-X(\alpha_{m-1}(t))| + \max_{t\in\mathbf{V}_{\varepsilon_m}}|X(t)-X(\alpha_{m-2}(t))|+\dots\\ &\quad + \max_{t\in\mathbf{V}_{\varepsilon_m}}|X(t)-X(\alpha(t_0))|+|X(t_0)|.\end{aligned}$$

Since

$$\sup_{t\in\mathbf{T}}|X(t)| = \sup_{t\in\mathbf{V}}|X(t)| \le |X(t_0)| + \sum_{l=1}^{m}\max_{t\in\mathbf{V}_{\varepsilon_l}}|X(t)-X(\alpha_{l-1}(t))|$$
$$\le |X(t_0)| + \sum_{l=1}^{\infty}\max_{t\in\mathbf{V}_{\varepsilon_l}}|X(t)-X(\alpha_{l-1}(t))|,$$

we conclude that

$$\left\|\sup_{t\in\mathbf{T}}|X(t)|\right\| \le \|X(t_0)\| + \sum_{l=1}^{\infty}\left\|\max_{t\in\mathbf{V}_{\varepsilon_l}}|X(t)-X(\alpha_{l-1}(t))|\right\|,$$

whence we deduce that

$$\left\|\sup_{t\in\mathbf{T}}|X(t)|\right\| \le \inf_{t\in\mathbf{T}}\|X(t)\| + \sum_{l=1}^{\infty}\varkappa(N(\varepsilon_l))\max_{t\in\mathbf{V}_{\varepsilon_l}}\|X(t)-X(\alpha_{l-1}(t))\|$$
$$\le \inf_{t\in\mathbf{T}}\|X(t)\| + \sum_{l=1}^{\infty}\varkappa(N(\varepsilon_l))\,\sigma(\varepsilon_l)$$
$$\le \inf_{t\in\mathbf{T}}\|X(t)\| + \sum_{l=1}^{\infty}\varkappa\left(N\left(\sigma^{(-1)}(\gamma p^k)\right)\right)\gamma p^{k-1}.$$

Next

$$\int_{\gamma p^{k+1}}^{\gamma p^k}\varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du \ge \varkappa\left(N\left(\sigma^{(-1)}(\gamma p^k)\right)\right)\left(\gamma p^k-\gamma p^{k+1}\right)$$
$$= \varkappa\left(N\left(\sigma^{(-1)}(\gamma p^k)\right)\right)\gamma p^{k-1}(p-p^2),$$

whence we establish that

$$\gamma p^k\varkappa\left(N\left(\sigma^{(-1)}(\gamma p^k)\right)\right) \le \frac{\int_{\gamma p^{k+1}}^{\gamma p^k}\varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du}{p(1-p)}.$$

Substituting this result in the preceding inequality, we get

$$\left\|\sup_{t\in\mathbf{T}}|X(t)|\right\| \le \inf_{t\in\mathbf{T}}\|X(t)\| + \sum_{l=1}^{\infty}\frac{1}{p(1-p)}\int_{\gamma p^{l+1}}^{\gamma p^l}\varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du$$
$$= \inf_{t\in\mathbf{T}}\|X(t)\| + \frac{1}{p(1-p)}\int_0^{\gamma p}\varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right)du. \qquad \square$$

Corollary 4.1. *Let $X=\{X(t), t\in\mathbf{T}\}$ be a stochastic process and assume that*

$$\sup_{t\in\mathbf{T}}|X(t)|\in\mathbf{F}_\psi^*(\Omega).$$

Then

$$\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|X(t)|>\varepsilon\right\} \le \inf_{u\ge1}\frac{B^u(p)(\psi(u))^u}{\varepsilon^u}$$

for all $\varepsilon>0$.

Example 4.1. If $\psi(u)=u^\alpha$ and $\alpha>0$, then

$$\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|X(t)|>x\right\} \le \exp\left\{-\frac{\alpha}{e}\left(\frac{x}{B(p)}\right)^{1/\alpha}\right\}$$

for $x\ge e^\alpha B(p)$.

Example 4.2. If $\psi(u) = e^{au}$ and $a > 0$, then

$$\mathsf{P}\{|\xi| > x\} \le \exp\left\{ -\frac{\left(\ln \frac{x}{B(p)}\right)^2}{4a} \right\}$$

for all $x \ge e^{2a} B(p)$.

Example 4.3. If $\psi(u) = e^{u^2}$, then

$$\mathsf{P}\left\{|\xi| > x\right\} \le \exp\left\{ -\frac{2\left(\ln \frac{x}{B(p)}\right)^{3/2}}{3^{3/2}} \right\}$$

for all $x \ge e^3 B(p)$.

Corollary 4.2. *Let $X = \{X(t), t \in [0,T]\}$, $T > 0$, be a separable stochastic process belonging to the space $\mathbf{F}^*_\psi(\Omega)$. Assume that*

$$\sup_{|t-s|\le h} \left\| X(t) - X(s) \right\| \le \sigma(h), \tag{24}$$

where $\sigma = \{\sigma(h), h > 0\}$ is a continuous increasing function such that $\sigma(0) = 0$. If

$$\int_0^z \varkappa\left(\frac{T}{2\sigma^{(-1)}(u)} + 1 \right) du < \infty$$

for all $z > 0$, then

$$\sup_{t\in[0,T]} |X(t)| \in \mathbf{F}^*_\psi(\Omega)$$

with probability one and

$$\left\| \sup_{t\in[0,T]} |X(t)| \right\| \le \widetilde{B}(p) \tag{25}$$

for all $0 < p < 1$, where

$$\widetilde{B}(p) = \inf_{t\in[0,T]} \|X(t)\| + \frac{1}{p(1-p)} \int_0^{\gamma p} \varkappa\left(\frac{T}{2\sigma^{(-1)}(u)} + 1 \right) du$$

and $\gamma = \sigma(T)$. Moreover,

$$\mathsf{P}\left\{ \sup_{t\in[0,T]} |X(t)| > \varepsilon \right\} \le \inf_{u\ge 1} \frac{\widetilde{B}^u(p)(\psi(u))^u}{\varepsilon^u} \tag{26}$$

for all $\varepsilon > 0$.

Proof. Corollary 4.2 follows from Theorem 4.1 since the metric capacity of the interval $[0,T]$ is such that

$$N(u) \le \frac{T}{2u} + 1.$$

□

Corollary 4.3. *Let $X = \{X(t), t \in [0,T]\}$, $T > 0$, be a separable stochastic process belonging to the space $\mathbf{F}^*_\psi(\Omega)$. Assume that*

$$\sup_{|t-s|\le h} \left\| X(t) - X(s) \right\| \le \frac{C}{\left(\varkappa\left(\frac{T}{2h} + 1\right)\right)^{1/\beta}} \tag{27}$$

for some $\beta < 1$. Then

$$\sup_{t\in[0,T]} |X(t)| \in \mathbf{F}^*_\psi(\Omega)$$

with probability one and

$$\left\|\sup_{t\in[0,T]}|X(t)|\right\| \le \inf_{t\in[0,T]}\|X(t)\| + \frac{C^{\beta}}{(1-\beta)}\left(\frac{C^{\beta}}{\varkappa\left(\frac{3}{2}\right)}\right)^{(1-\beta)/\beta}\frac{(1+\beta)^{\beta+1}}{\beta^{\beta}} = \widetilde{B}.$$

Moreover,

$$\mathsf{P}\left\{\sup_{t\in[0,T]}|X(t)|>\varepsilon\right\} \le \inf_{u\ge 1}\frac{\widetilde{B}^{u}(\psi(u))^{u}}{\varepsilon^{u}} \tag{28}$$

for all $\varepsilon > 0$.

Proof. Corollary 4.3 follows from Corollary 4.2. Indeed,

$$\sigma(h) = \frac{C}{\left(\varkappa\left(\frac{T}{2h}+1\right)\right)^{1/\beta}}$$

and thus

$$\frac{1}{p(1-p)}\int_0^{\gamma p}\varkappa\left(\frac{T}{2\sigma^{(-1)}(u)}+1\right)du = \frac{1}{p(1-p)}\int_0^{\gamma p}\frac{C^{\beta}}{u^{\beta}}\,du = \frac{C^{\beta}}{1-\beta}\gamma^{1-\beta}\frac{1}{(1-p)p^{\beta}}.$$

Minimizing the right hand side of the latter inequality with respect to p, we derive Corollary 4.3 from inequality (25). □

Set

$$B(\beta) = \frac{C^{\beta}}{(1-\beta)}\left(\frac{C^{\beta}}{\varkappa\left(\frac{3}{2}\right)}\right)^{(1-\beta)/\beta}\frac{(1+\beta)^{\beta+1}}{\beta^{\beta}}.$$

Example 4.4. Consider the space $\mathbf{F}_{\psi}(\Omega)$ for $\psi(u) = u^{\alpha}$. Put

$$\sigma(h) = \frac{C}{\left(\ln\left(\frac{T}{2h}+1\right)\frac{e}{\alpha}\right)^{\alpha/\beta}}.$$

Then

$$\left\|\sup_{t\in[0,T]}|X(t)|\right\| \le \inf_{0\le t\le T}\sup_{u\ge 1}\frac{\left(\mathsf{E}\,|X(t)|^{u}\right)^{1/u}}{u^{\alpha}} + B(\beta) = B_{u^{\alpha}}.$$

Moreover,

$$\mathsf{P}\left\{\sup_{t\in[0,T]}|X(t)|>x\right\} \le \exp\left\{-\frac{\alpha}{e}\left(\frac{x}{B_{u^{\alpha}}}\right)^{1/\alpha}\right\}$$

for all $x > 0$.

Example 4.5. Consider the space $\mathbf{F}_{\psi}(\Omega)$ for $\psi(u) = e^{au}$, where $a > 0$. Put

$$\sigma(h) = \frac{C}{\exp\left\{\left(2\sqrt{a\ln\left(\frac{T}{2h}+1\right)}-a\right)\frac{1}{\beta}\right\}}$$

and

$$\left\|\sup_{t\in[0,T]}|X(t)|\right\| \le \inf_{0\le t\le T}\sup_{u\ge 1}\frac{\left(\mathsf{E}\,|X(t)|^{u}\right)^{1/u}}{e^{au}} + B(\beta) = B_{e^{au}}.$$

Then

$$\mathsf{P}\left\{\sup_{t\in[0,T]}|X(t)|>x\right\} \le \exp\left\{-\frac{\left(\ln\frac{x}{B_{e^{au}}}\right)^{2}}{4a}\right\}$$

for all $x > e^{2a}B_{e^{au}}$.

5. Concluding remarks

Some properties of random variables and stochastic processes belonging to the spaces $\mathbf{F}_\psi(\Omega)$ are studied in this paper. In a forthcoming publication, we plan to apply the results obtained in the current paper to the Monte-Carlo method for the evaluation of multiple integrals with a given accuracy.

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